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Monotone approximations of unstable solutions

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Abstract

Assume that for a nonlinear two-point problem we have a subsolution lying above a supersolution. It is well-known that in general one cannot conclude the existence of a solution in between. However, if one restricts the growth of the nonlinearity, then under some restrictions on the super- and subsolution, one gets both existence of a solution and two sequences of monotone approximations. In addition, we can assert some qualitative property of the solution. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is well-known that if a nonlinear problem

$$u''(x) + f(x,u(x)) = 0 \quad \text{for } x \in (0,L), \quad u(0) = u(L) = 0, \tag{1}$$

has a subsolution $\psi(x)$ and a supersolution $\phi(x)$, which are ordered, i.e., $\psi(x) < \phi(x)$ for all $x \in (0,L)$, then there exist at least one solution $u(x)$ with $\psi(x) < u(x) < \phi(x)$. Moreover, one can construct two sequences of approximations converging, respectively, to minimal and maximal solutions of (1), see, e.g., [1]. This approach works only for stable solutions of (1). Recall that a solution $u(x)$ of (1) is called stable if all eigenvalues λ of the corresponding linearized problem

$$w''(x) + f_u(x,u(x))w(x) = \lambda w(x) \quad \text{for } x \in (0,L), \quad w(0) = w(L) = 0$$

are negative. It is well known that for an unstable solution it is impossible to find a subsolution $\psi(x)$ and a supersolution $\phi(x)$, so that $\psi(x) < u(x) < \phi(x)$, see, e.g., [4] for the discussion.

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It is natural to ask whether we still have existence of solutions if the subsolution $\psi(x)$ lies above the supersolution $\phi(x)$, i.e.,

$$\psi(x) > \phi(x) \quad \text{for all } x \in (0,L). \tag{2}$$

It has been known for a while that in general this is not true. The standard counterexample due to Kazdan and Warner, see [1, p. 653], involves $f(x,u)=\lambda_2 u+v(x)$, where λ_2 is the second eigenvalue of u'' on $(0,L)$ with zero boundary conditions, and $v(x)$ is a suitably constructed function. (λ_2 may be replaced by any λ_k with $k \geqslant 2$.) We wish to raise the following question: what if we assume (2), and add a condition

$$f_u(x,u) < \lambda_2. \tag{3}$$

Can we conclude that problem (1) has a solution? We cannot answer this question, but we present an example indicating that some general results in this direction might be possible.

Our main result is given by the following theorem. It uses the concept of front-loaded on $(0,L/2)$ functions (or just "front-loaded functions", for short), which is defined in the next section. Increasing on $(0,L/2)$ functions provide a simple example of front-loaded functions. We shall work with even with respect to $x=L/2$ functions, or "even functions" for short.

Theorem 1. *Assume that the problem* (1) *has a strict front-loaded subsolution, i.e., a function* $\psi(x) \in C^2(0,L) \cap C[0,L]$ *satisfying*

$$\psi''(x) + f(x,\psi(x)) = \bar{h}(x) \quad for\ x \in (0,L), \quad \psi(0)=\psi(L)=0, \tag{4}$$

where $\bar{h}(x)$ *is a positive front-loaded on* $(0,L/2)$ *function, and a strict front-loaded supersolution* $\phi(x) \in C^2(0,L) \cap C[0,L]$, *satisfying*

$$\phi''(x) + f(x,\phi(x)) = -h(x) \quad for\ x \in (0,L), \quad \phi(0)=\phi(L)=0 \tag{5}$$

with $h(x)$ *being positive and front-loaded, and* $\psi(x) > \phi(x)$ *for all* $x \in (0,L)$. *Assume that* $\phi(x)$, $\psi(x)$ *and* $\psi(x)-\phi(x)$ *are even and front-loaded functions. Assume that the function* $f(x,u)$ *is twice differentiable in both arguments, and even in* x *with respect to* $x=L/2$, *for all* $x \in (0,L)$ *and* $\phi(x)<u<\psi(x)$. *Assume finally that inequality* (3) *and* $f_{uu}(x,u)<0$ *hold, and that the function* $-f_u(x,u)$ *is front-loaded in* x, *for all* $x\in(0,L)$ *and* $\phi(x)<u<\psi(x)$. *Then problem* (1) *admits a classical solution. Moreover, one can construct two sequences of monotone iterates*

$$\phi < u_1 < u_2 < \cdots < u_n < \cdots < v_k < \cdots < v_2 < v_1 < \psi,$$

converging to even front-loaded solutions of (1) (*possibly to the same one*).

We remark that we call the super- and subsolution strict, since we require them to satisfy the Dirichlet boundary conditions (rather than the corresponding inequalities). Multiples of the principal eigenfunction will typically provide strict front-loaded super- and subsolutions. The main restriction of the above theorem, the concavity of $f(x,u)$ in u, was used only once in the proof, and we suspect it can be dropped (which is supported by our numerical experiments). We also note that condition (3) has appeared previously in Lazer and McKenna [6], where the authors noticed that this condition simplifies the solution structure for the PDE version of (1).

Our main tool is an anti-maximum principle at the second eigenvalue, which appears to be also of independent interest. Starting with Clement and Peletier [3], a number of anti-maximum principles have appeared in the literature. They usually hold slightly above the first eigenvalue. We obtain an anti-maximum principle at λ_2 by restricting to even in x equations, and their even solutions.

2. An anti-maximum principle

Without loss of generality, we may consider our equations on the interval $(0,L)=(0,\pi)$, so that we have $\lambda_1=1$ and $\lambda_2=4$.

Definition. We say that a function $f(x)\in C[0,\pi/2]$ is front-loaded, or of class FL, if

$$f\left(\frac{\pi}{4}+t\right)\geqslant f\left(\frac{\pi}{4}-t\right)\quad \text{for all } t\in[0,\pi/4], \tag{6}$$

with the inequality being strict on a set of positive measure.

Lemma 1. *A function $f(x)\in C[0,\pi/2]$, is front-loaded if and only if we can write it as a sum of two continuous functions, $f(x)=f_e(x)+g(x)$, where $f_e(x)$ is even with respect to $\pi/4$, and $g(x)=0$ for $x\in(0,\pi/4)$, while $g(x)\geqslant 0$ for $x\in(\pi/4,\pi/2)$. If $f(x)$ is positive on $(0,\pi/2)$, so is $f_e(x)$.*

Proof. Just define $f_e(x)$ to be an even extension of $f(x)$ from $(0,\pi/4)$ to $(0,\pi/2)$, and use (6). □

Clearly, the class FL is closed under addition, multiplication (if both functions are positive) and integration with respect to a parameter. The following lemma is also obvious.

Lemma 2. *Assume $f(x)>0$ is of class* FL *on $(0,\pi/2)$. Then*

$$\int_0^{\pi/2} f(\xi)\cos 2\xi \,\mathrm{d}\xi<0. \tag{7}$$

The following lemma is a well-known version of maximum principle. For completeness we sketch a proof. Observe that on the interval $(0,\pi/2)$ the principal eigenvalue of $-u''$ is $\lambda_1=4$.

Lemma 3. *Consider the problem*

$$u''+\lambda u=f(x)\quad \textit{on } (0,\pi/2),\quad u(0)=u(\pi/2)=0. \tag{8}$$

Assume that $f(x)$ is continuous and $f(x)\geqslant 0$ on $(0,\pi/2)$, and $\lambda<4$. Then $u(x)\leqslant 0$ on $(0,\pi/2)$.

Proof. Assume on the contrary that $u>0$ on some subinterval $(\alpha,\beta)\subset(0,\pi/2)$. Multiplying (8) by $\sin(\pi/(\beta-\alpha))(x-\alpha)$, and integrating over (α,β), we obtain a contradiction. □

Definition. We say that a function $f(x,u)\in C([0,\pi/2]\times(u_1,u_2)$ is front-loaded with respect to x if

$$f\left(\frac{\pi}{4}+t,u\right)\geqslant f\left(\frac{\pi}{4}-t,u\right)\quad \text{for all } t\in[0,\pi/4],\quad u\in(u_1,u_2).$$

(Notice that in contrast to the class FL, the inequality sign need not be strict here. In particular, the case of $f=f(u)$ is included.)

We shall need the following lemma in the next section. We omit its simple proof.

Lemma 4. *Assume that a function $f(x,u)$ is front-loaded with respect to x, and increasing in u. Assume that $u(x)$ is of class* FL. *Then $f(x,u)$ is of class* FL.

We present the anti-maximum principle next.

Theorem 2. *Consider the problem*

$$u'' + 4u = f(x) \quad \text{on } (0,\pi), \quad u(0) = u(\pi) = 0. \tag{9}$$

Assume that $f(x)$ is continuous, even with respect to $x=\pi/2$, of class FL *on $(0,\pi/2)$, and $f(x) > 0$ on $(0,\pi)$. Then there is a solution $u(x)$, which is also even with respect to $x = \pi/2$ and positive. Moreover, $u(x)$ is of class* FL on $(0,\pi/2)$.

Proof. The kernel of (9) is spanned by $\sin 2x$. Since $f(x)$ is even, we have

$$\int_0^{\pi} f(\xi)\sin 2\xi \, d\xi = 0. \tag{10}$$

Using (10) we conclude that the general solution of (9) is given by

$$u(x) = \frac{1}{2}\int_0^x \sin 2(x-\xi) f(\xi)\, d\xi + c\sin 2x, \tag{11}$$

where c is an arbitrary constant. To obtain an even solution, we set

$$c_0 = \int_0^{\pi/2} f(\xi)\cos 2\xi \, d\xi = \frac{1}{2}\int_0^{\pi} f(\xi)\cos 2\xi \, d\xi, \tag{12}$$

and select a particular solution

$$u_0(x) = \frac{1}{2}\int_0^x \sin 2(x-\xi) f(\xi)\, d\xi - \frac{1}{2}c_0 \sin 2x. \tag{13}$$

To see that $u_0(x)$ is even with respect to $\pi/2$, we rewrite it as

$$u_0(x) = -\frac{1}{2}\sin 2x \int_x^{\pi/2} f(\xi)\cos 2\xi \, d\xi - \frac{1}{2}\cos 2x \int_0^x f(\xi)\sin 2\xi \, d\xi. \tag{14}$$

The first term on the right is the product of two odd with respect to $\pi/2$ functions, while the second one is a product of two even functions. It follows that $u_0(x)$ is even with respect to $\pi/2$.

Since $f(x)$ is front-loaded, it follows by Lemma 2 that

$$u_0'(0) = -c_0 > 0.$$

It follows that $u_0(x)$ is positive for small $x > 0$. We proceed to show that it is positive for all $x \in (0,\pi)$. Indeed, since $u_0(x)$ is even, if it were to vanish on $(0,\pi)$, it would have to have its first (the smallest) root ξ on the left half of the interval, i.e. $\xi \in (0,\pi/2]$. But $\lambda_2 = 4$ is the principal eigenvalue of the half-interval $(0,\pi/2)$. If $\xi=\pi/2$, we would have a solution of the Dirichlet problem on $(0,\pi/2)$, with the positive right-hand side $f(x)$, which cannot be orthogonal to the kernel, spanned by the principal (positive) eigenfunction $\sin 2x$, a contradiction. If $\xi \in (0,\pi/2)$, then by Lemma 3 we cannot have a positive solution of Dirichlet problem for (9) when the right-hand side is positive, a contradiction. (The principal eigenvalue of $-u''$ is greater than 4 on $(0,\xi)$.)

It remains to show that $u_0(x)$ is front-loaded on $(0,\pi/2)$. On that interval, the second term in (13) is even with respect to $\pi/4$, so it suffices to show that,

$$I \equiv \int_0^x \sin 2(x-\xi)f(\xi)\,\mathrm{d}\xi$$

is front-loaded. Using the decomposition of Lemma 1 we write

$$I = \int_0^x \sin 2(x-\xi)f_{\mathrm{e}}(\xi)\,\mathrm{d}\xi + \int_0^x \sin 2(x-\xi)g(\xi)\,\mathrm{d}\xi \equiv I_1 + I_2.$$

Clearly I_2 is front-loaded, since it is equal to zero on $(0,\pi/4)$, and it is positive on $(\pi/4,\pi/2)$. Write

$$I_1 = \sin 2x \int_0^x \cos 2\xi f_{\mathrm{e}}(\xi)\,\mathrm{d}\xi - \cos 2x \int_0^x \sin 2\xi f_{\mathrm{e}}(\xi)\,\mathrm{d}\xi \equiv I_{11} + I_{12}.$$

The term I_{11} is even with respect to $\pi/4$, as a product of two even with respect to $\pi/4$ functions. The term I_{12} is front-loaded, since it is negative on $(0,\pi/4)$, and it is positive on $(\pi/4,\pi/2)$. The proof follows.

Remarks. 1. In case $f(0)>0$, we may allow the equal sign in the definition of the front-loaded function (6). Then we have $u_0'(0)=0$, but $u_0''(0)=f(0)>0$, and so $u_0(x)$ starts out positive, as before. In particular, the theorem still holds if $f(x)$ is a positive constant (even though it is not of class FL). This will allow us to extend our main result to the case of either $h(x)$ or $\bar{h}(x)$ (in the definition of sub- and supersolution) being a positive constant.

2. Instead of assuming that $f(x)$ is of class FL, we could assume that $f(x)$ satisfies a more general condition (7). It is easy to see that the anti-maximum principle holds, and in addition, $u(x)$ also satisfies (7). However, property (7) does not seem to be preserved under composition of functions, and so this more general version of anti-maximum principle could not be used for the nonlinear problems.

3. If we only assume $f(x)$ to be even with respect to $\pi/2$, we can still assert that (9) has an even with respect to $\pi/2$ solution (it is given by formula (14)).

3. Proof of the main result

Setting $u_0(x)=\phi(x)$, we define inductively a sequence $u_n(x)$ by solving for $n=0,1,\ldots$

$$u''_{n+1} + \lambda_2 u_{n+1} = \lambda_2 u_n - f(x,u_n) \quad \text{for } x \in (0,L),$$
$$u_{n+1}(0) = u_{n+1}(L) = 0. \tag{15}$$

When $n=0$ the right-hand side of (15) $g(x) \equiv \lambda_2\phi - f(\phi)$ is an even with respect to $x=L/2$ function, and hence $\int_0^L g(x)\phi_2(x)\,\mathrm{d}x = 0$, and so problem (15) is solvable. (Here $\phi_2 = \sin(2\pi/L)x$, the second eigenfunction.) Since $\lambda_2\phi - f(x,\phi)$ is even, by Remark 3 to Theorem 2 we can select $u_1(x)$ to be even, and continuing inductively, we select u_{n+1} to be the even solution of problem (15). By the definition of the supersolution, there is a positive even front-loaded function $h(x)$, so that

$$\phi'' + \lambda_2\phi = \lambda_2\phi - f(x,\phi) - h(x).$$

(We did not assume explicitly that $h(x)$ is even, but this follows from our assumptions that $\phi(x)$ and $f(x,u)$ are even in x.) Combining that with the case of $n=0$ in (15), we have

$$(u_1-\phi)''+\lambda_2(u_1-\phi)=h(x)>0 \quad \text{on } (0,L), \quad (u_1-\phi)(0)=(u_1-\phi)(L)=0.$$

By the anti-maximum principle, Theorem 2, $u_1(x)>\phi(x)$ for all x, and the function $u_1-\phi$ is front-loaded. It follows that $u_1=\phi+u_1-\phi$ is also front-loaded. Proceeding inductively, we have

$$\begin{aligned}&(u_{n+1}-u_n)''+\lambda_2(u_{n+1}-u_n)=g(x,u_n)-g(x,u_{n-1}) \quad \text{on } (0,L),\\ &(u_{n+1}-u_n)(0)=(u_{n+1}-u_n)(L)=0.\end{aligned} \tag{16}$$

Notice that the function $g(x,u)\equiv\lambda_2u-f(x,u)$ is increasing in u for all x, and for u between super- and subsolution. By the inductive assumption u_n-u_{n-1} is a positive front-loaded function, and the function u_n is front-loaded. It follows that the right-hand side of (16) is positive. To see that it is also front-loaded, we write

$$g(x,u_n)-g(x,u_{n-1})=\int_0^1 g_u(x,\theta u_n+(1-\theta)u_{n-1})\,\mathrm{d}\theta(u_n-u_{n-1}). \tag{17}$$

Since by our assumptions $g_u(x,u)$ is front-loaded in x, and $(\mathrm{d}/\mathrm{d}u)g_u=-f_{uu}>0$, it follows by Lemma 4 that $g_u(x,\theta u_n+(1-\theta)u_{n-1})$ is a positive front-loaded function, and hence the right-hand side of (17) is a front-loaded function. Applying the anti-maximum principle to (16), we conclude that $u_{n+1}-u_n$ is positive and front-loaded. Hence u_{n+1} is also front-loaded and the sequence $\{u_n(x)\}$ is increasing. To see that all elements of this sequence lie below the subsolution ψ, we write

$$\begin{aligned}&(\psi-u_1)''+\lambda_2(\psi-u_1)=g(x,\psi)-g(x,\phi)+\bar{h}(x) \quad \text{on } (0,L),\\ &(\psi-u_1)(0)=(\psi-u_1)(L)=0,\end{aligned} \tag{18}$$

where $\bar{h}(x)$ is positive, even and front-loaded. Arguing as in (17), and using that $\psi-\phi$ is a positive front-loaded function, we conclude that the right-hand side of (18) is also a positive front-loaded function, and using the anti-maximum principle once more, we have $u_1<\psi$ (and also that $\psi-u_1$ is of class FL). Proceeding inductively, we have

$$\phi<u_1<u_2<\cdots<\psi.$$

We then define another sequence $v_n(x)$ by solving (15), starting with $v_0(x)=\psi(x)$, to obtain

$$\phi<u_1<u_2<\cdots<v_2<v_1<\psi. \tag{19}$$

From (19) we see that both the sequences $u_n(x)$ and $v_n(x)$ converge, and passing to the limit in the integral equation formulation of (15), given by (11), we obtain solution(s) of our problem.

Example 1. Consider the problem

$$u''+\lambda(\sqrt{u+1}-2)=0 \quad \text{for } x\in(0,L), \quad u(0)=u(L)=0. \tag{20}$$

Here $f(u)=\lambda(\sqrt{u+1}-2)$ is a concave function with $f(0)<0$, λ is a positive parameter. It is known that there exist $0<\mu_1<\mu_2$ so that the problem (20) has no positive solution for $\lambda<\mu_1$, exactly one for $\lambda=\mu_1$ and $\lambda>\mu_2$, and exactly two for $\mu_1<\lambda<\mu_2$, see [2] or [5]. Problem (20) has a supersolution $\phi=0$, which can be used to compute the minimal positive solution for $\lambda>\mu_1$. Numerical computations confirmed the validity of the scheme.

Positive subsolutions of (20) certainly exist, for example any positive solution of a similar problem

$$v'' + \lambda(\sqrt{v+1} - 3) = 0 \quad \text{for } x \in (0, L), \quad v(0) = v(L) = 0$$

is a subsolution of (20). However, it is not easy to construct an explicit subsolution. This prompts us to remark that while our Theorem 2 provides a computational tool, with novel "reverse monotone iterations", it does not appear to be very effective for proving existence of solutions.

Example 2. The problem

$$u'' + u^2 = 0 \quad \text{for } x \in (0, 1), \quad u(0) = u(1) = 0 \tag{21}$$

is not covered by our result, since the nonlinearity is not concave. However in our numerical computation the scheme still worked. Namely, we started with a supersolution $\phi(x) = 0.1 \sin \pi x$ (which is even and front-loaded), and obtained a sequence of monotone increasing iterates, which converged quickly to the unique positive solution of (21).

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